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The exciting force on a submerged spheroid in regular waves

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The hydrodynamic problem of a submerged spheroid in waves is analysed based on linearized potential theory. An analytic formulation is derived and demonstrated by considering the problem of a stationary spheroid in head or following seas. Tabulated numerical results are obtained for a spheroid whose major axis is six times the minor axis, submerged at a depth twice the minor axis. Figures for many other cases are also provided. It is suggested that the present method can be extended to the problem of oscillating bodies at forward speed.

1. Introduction

In the prediction of motions of ships and offshore structures in waves, the structure is usually regarded as a rigid body having six degrees of freedom. The fluid loading on the structure is estimated by linearized potential theory, assuming that the fluid is inviscid and incompressible, the fluid flow is irrotational, and both the incoming wave elevation and the body oscillation are small. The velocity potential ϕ then satisfies Laplace's equation, and the linearized conditions are adopted on the mean position of the fluid boundary.

For most practical geometries, this boundary-value problem can be solved only by a numerical method. The approach usually followed is to find a function which satisfies the governing equation at discretized points or in a uniform sense rather than in the whole continuous fluid domain and its boundary. In principle, with such a method one can achieve any desired accuracy by imposing the governing equation at more and more discretized points. But this process has obvious practical limitations such as computer time. It is not surprising therefore that different computer programs using different numerical methods sometimes do not give the same results (Eatock Taylor & Jefferys 1986). In these circumstances, part of the process of validating the numerical method and establishing convergence characteristics is to make comparison with analytical solutions.

Because of the complexity of ship hydrodynamics problems, analytical solutions can only be obtained for a few special geometries. Havelock analysed a floating sphere in heave motion in 1955. His analysis was extended by Hulme (1982) to the sphere undergoing both vertical and horizontal motions, and very recently Wang (1986) investigated both radiation and diffraction problems for a submerged sphere. These studies are all based on the method of multipole expansions (Thorne 1953), which has been proved to be very successful for periodic motions without forward speed.

It seems difficult to extend this method to the problem of a body with forward speed. Thus as a first step to solving the forward-speed problem, this work follows a rather different procedure. We use the traditional source-distribution method; but instead of dividing the body surface into many small panels, as in many numerical methods, we expand the source strength into a series of Legendre functions. This approach was originally used by Farell (1973) in analysing the wave resistance on a submerged spheroid having constant forward speed in otherwise calm sea. His formulations have been extended to the case of a submerged spheroid advancing in regular waves by Wu (1986), and this paper concerns the special case of zero forward speed. The results may be used as a partial check on the more difficult analysis with forward speed. Furthermore the present formulation can provide accurate results for a wider range of body geometries than circular cylinders and spheres. It has been observed that while many computer programs give very similar results for a very simple geometry such as a sphere, they fail to reach agreement for many practical structures. This work considers the problem of a submarine-like body, namely a spheroid. Such a geometry can cause difficulties with some of the numerical methods if the aspect ratio is very large: the problem is associated with adopting a sufficiently fine mesh near the sharp ends. The analytical solution procedure, however, should be capable of resolving the ideal flow in such cases. Results are obtained here for the exciting forces on a submerged spheroid in head or following seas. These cases allow symmetry to be used in the analysis, although the assumption regarding the incoming-wave direction is not essential to the method of analysis.

2. The mathematical formulation

We define the coordinates $Oxyz$ so that z points vertically upwards with $z = 0$ being the mean free surface, Ox is parallel to the major axis of the spheroid, and the coordinates follow the right-hand rule. The centre of the spheroid is located at $(0, 0, -h)$. A system of spheroidal coordinates (η, θ, ϕ) is defined by

$$x = c \cosh \eta \cos \theta, \tag{1a}$$

$$y = c \sinh \eta \sin \theta \sin \phi, \tag{1b}$$

$$z = c \sinh \eta \sin \theta \cos \phi - h. \tag{1c}$$

When $\eta = \eta_0$, (1) defines the surface of a spheroid having major axis $a = c \cosh \eta_0$ and minor axis $b = c \sinh \eta_0$.

For harmonic fluid motions, the time-dependent velocity potential Φ can be taken as $\Phi = \mathrm{Re}(\phi\, \mathrm{e}^{\mathrm{i}\omega t})$. Based on the assumption of the linearized theory, the time-independent potential ϕ will satisfy the following equations

$$\nabla^2 \phi = 0 \quad \text{in the fluid}, \tag{2}$$

$$\frac{\partial \phi}{\partial z} - \nu \phi = 0 \quad \text{on } z = 0, \tag{3}$$

$$R^{\frac{1}{2}} \left(\frac{\partial}{\partial R} - \mathrm{i}\nu \right) \phi \rightarrow 0 \quad R \rightarrow \infty, \tag{4}$$

where $\nu = \omega^2/g$ is the wavenumber, ω is the wave frequency, g is the gravitational acceleration and $R = (x^2 + y^2)^{\frac{1}{2}}$. The water depth is assumed to be infinite. For the diffraction problem, the potential ϕ_d also satisfies the body surface condition

$$\frac{\partial \phi_\mathrm{d}}{\partial n} = -\frac{\partial \phi_\mathrm{I}}{\partial n}, \tag{5}$$

where n is the normal of the body surface S_0 and ϕ_i is the incident potential which for a wave of unit amplitude can be written as

$$\phi_i = \frac{ig}{\omega} \exp\left[\nu z - i\nu\, (x\cos\beta + y\sin\beta)\right]. \tag{6}$$

The incoming-wave incident angle is defined so that $\beta = 0$ corresponds to a following sea and $\beta = \pi$ corresponds to a head sea.

The unknown potential (in this case ϕ_d) can be represented by a source distribution $\sigma(\chi, \eta, \zeta)$ over the body surface

$$\phi_d = \iint_{S_0} \sigma(\chi, \eta, \zeta)\, G(x, y, z, \chi, \eta, \zeta)\, dS, \tag{7}$$

where G is the well-known Green function which can be written as (Wehausen & Laitone 1960)

$$G = \frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi}\int_{-\pi}^{\pi}\int_L \frac{1}{k-\nu} \exp\{k(z+\zeta) + ik[(x-\chi)\cos t + (y-\eta)\sin t]\}\, dk\, dt, \tag{8}$$

$$r = ((x-\chi)^2 + (y-\eta)^2 + (z-\zeta)^2)^{\frac{1}{2}}, \tag{9a}$$

$$r_1 = ((x-\chi)^2 + (y-\eta)^2 + (z+\zeta)^2)^{\frac{1}{2}}, \tag{9b}$$

and the integration path L is from 0 to ∞ and overpasses the singularity at $k = \nu$.

The potential ϕ_d expressed by (7) satisfies all governing equations except the body-surface condition. This condition is met by an appropriate choice of the source distribution, which can be achieved by using

$$\frac{\partial \phi}{\partial n} = 2\pi\sigma + \iint_{S_0} \sigma \frac{\partial G}{\partial n}\, dS. \tag{10}$$

Since the left-hand side is known from (5), the solution for the source distribution can be obtained.

For an arbitrary body, the solution of (10) requires discretization of the body surface and the problem has to be solved numerically. For a submerged spheroid, the source distribution can be expanded as a series of Legendre functions (Farell 1973), or

$$\sigma = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m \frac{(-1)^{m+1}}{4\pi} \frac{P_n^m(\cos\theta)\cos m\phi}{P_n^m(\cosh\eta_0)\sinh\eta_0\,(\cosh^2\eta_0 - \cos^2\theta)^{\frac{1}{2}}}. \tag{11}$$

Thus substituting (8) and (11) into (7), we obtain

$$\begin{aligned}\phi_d = \iint_{S_0} \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m \frac{(-1)^{m+1}}{4\pi} &\frac{P_n^m(\cos\theta)\cos m\phi}{P_n^m(\cosh\eta_0)\sinh\eta_0\,(\cosh^2\eta_0 - \cos^2\theta)^{\frac{1}{2}}} \\ &\times\left\{\frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi}\int_{-\pi}^{\pi} dt \int_L dk \frac{1}{k-\nu}\exp\{k(z+\zeta) + ik[(x-\chi)\cos t + (y-\eta)\sin t)]\}\right\} dS.\end{aligned} \tag{12}$$

Use is made of the following relations (Farell 1973),

$$\iint_{S_0} \frac{1}{r}\,\sigma\,\mathrm{d}S = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m \frac{c(n-m)!}{(n+m)!} P_n^m(\cos\theta)\cos m\phi\, Q_n^m(\cosh\eta), \tag{13}$$

$$\iint_{S_0} \exp\left[k\zeta - \mathrm{i}k(\chi\cos t+\eta\sin t)\right]\sigma\,\mathrm{d}S = \tfrac{1}{2}c^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{n+1} \times \mathrm{i}^{n+m} A_n^m\left[(\sec t+\tan t)^m + \frac{1}{(\sec t+\tan t)^m}\right] j_n(\Delta)\exp(-kh), \tag{14}$$

$$\Delta = kc\cos t, \tag{15}$$

$$\exp\left[kz+\mathrm{i}k(x\cos t+y\sin t)\right] = \exp(-kh)\left[\sum_{n=0}^{\infty} C_{n,0}P_n(\cos\theta)P_n(\cosh\eta) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(C_{n,m}\cos m\phi + B_{n,m}\sin m\phi)P_n^m(\cos\theta)P_n^m(\cosh\eta)\right], \tag{16}$$

where

$$C_{n,0} = \mathrm{i}^n(2n+1)j_n(\Delta), \tag{17a}$$

$$C_{n,m} = \mathrm{i}^{n+m}\frac{(n-m)!}{(n+m)!}(2n+1)\left[(\sec t+\tan t)^m + \frac{1}{(\sec t+\tan t)^m}\right] j_n(\Delta) \quad (m \geqslant 1), \tag{17b}$$

$$B_{n,m} = \mathrm{i}^{n+m+1}\frac{(n-m)!}{(n+m)!}(2n+1)\left[(\sec t+\tan t)^m - \frac{1}{(\sec t+\tan t)^m}\right] j_n(\Delta), \tag{17c}$$

and j_n is the spherical Bessel function of the first kind. P_n^m and Q_n^m are the associated Legendre functions of the first and second kind respectively. We also use

$$\begin{aligned}\iint_{S_0}\frac{1}{r_1}\sigma\,\mathrm{d}S &= \iint_{S_0}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}t\int_0^{\infty}\mathrm{d}k\,\exp\{k(z+\zeta)+\mathrm{i}k[(x-\chi)\cos t+(y-\eta)\sin t]\}\,\sigma\,\mathrm{d}S \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}t\int_0^{\infty}\mathrm{d}k\,\exp(-2kh)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{n} C_{n,m}\cos m\phi P_n^m(\cos\theta)P_n^m(\cosh\eta)\right\} \\ &\quad\times\left\{\tfrac{1}{2}c^2\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}(-1)^{n'+1}\mathrm{i}^{n'+m'}A_{n'}^{m'}\left[(\sec t+\tan t)^{m'}+\frac{1}{(\sec t+\tan t)^{m'}}\right]j_{n'}(\Delta)\right\},\end{aligned} \tag{18}$$

to obtain

$$\begin{aligned}\phi = &-c\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m\frac{(n-m)!}{(n+m)!}Q_n^m(\cosh\eta)P_n^m(\cos\theta)\cos m\phi \\ &+\frac{c^2}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{P_n^m(\cosh\eta)P_n^m(\cos\theta)\cos m\phi\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n'+1}\mathrm{i}^{n'+m'}\right. \\ &\qquad\left.\times\left[\int_{-\pi}^{\pi}\mathrm{d}t\int_0^{\infty}\mathrm{d}k\,\exp(-2kh)\,T_{n'}^{m'}(t)j_{n'}(\Delta)C_{n,m}\right]\right\} \\ &+\frac{c^2\nu}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{P_n^m(\cosh\eta)P_n^m(\cos\theta)\cos m\phi\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n'+1}\mathrm{i}^{n'+m'}\right. \\ &\qquad\left.\times\left[\int_{-\pi}^{\pi}\mathrm{d}t\int_L\mathrm{d}k\,\frac{\mathrm{e}^{-2kh}}{k-\nu}\,T_{n'}^{m'}(t)j_{n'}(\Delta)C_{n,m}\right]\right\},\end{aligned} \tag{19}$$

where
$$T_n^m(t) = (\sec t+\tan t)^m + \frac{1}{(\sec t+\tan t)^m}. \tag{20}$$

The terms in $\sin m\phi$ have been neglected due to symmetry in the case of head or following seas. To impose the body-surface condition, we also write (6) in terms of Legendre functions as

$$\phi_{\mathrm{i}} = \frac{\mathrm{i}g}{\omega}\exp(-\nu h)\sum_{n=0}^{\infty}\sum_{m=0}^{n} D_{n,m}(\nu,\beta)\,P_n^m(\cosh\eta)\,P_n^m(\cos\theta)\cos m\phi, \tag{21}$$

with $\beta = 0$ or $\beta = \pi$, where

$$D_{n,m}(k,t) = C_{n,m}(k,t+\pi) = (-1)^{n+m}\,C_{n,m}(k,t). \tag{22}$$

Using the body surface condition

$$\frac{\partial\phi_{\mathrm{d}}}{\partial\eta} = -\frac{\partial\phi_{\mathrm{i}}}{\partial\eta},$$

on $\eta = \eta_0$, we obtain the infinite set of equations

$$\begin{aligned}
-\frac{2\mathrm{i}g}{\omega}&\exp(-\nu h)(-\mathrm{i})^{n+m}\gamma\frac{\mathrm{d}}{\mathrm{d}\eta}[P_n^m(\cosh\eta_0)]j_n(\nu c\cos\beta)\\
&= \frac{-c}{(2n+1)\alpha}A_n^m\frac{\mathrm{d}}{\mathrm{d}\eta}[Q_n^m(\cosh\eta_0)]\\
&\quad+\frac{c^2}{4\pi}\mathrm{i}^{n+m}\frac{\mathrm{d}}{\mathrm{d}\eta}[P_n^m(\cosh\eta_0)]\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n'+1}\,\mathrm{i}^{n'+m'}I(n,m,n',m')\\
&\quad+\frac{c^2\nu}{2\pi}\mathrm{i}^{n+m}\frac{\mathrm{d}}{\mathrm{d}\eta}[P_n^m(\cosh\eta_0)]\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n'+1}\,\mathrm{i}^{n'+m'}H(n,m,n',m'),
\end{aligned} \tag{23}$$

where $\alpha = \frac{1}{2}$ for $m = 0$, $\alpha = 1$ for $m > 0$; and $\gamma = 1$ for $\beta = 0$, $\gamma = (-1)^m$ for $\beta = \pi$. The two integrals I and H in equation (23) are defined by

$$I(n,m,n',m') = \int_{-\pi}^{\pi} T_n^m(t)\,T_{n'}^{m'}(t)\,\mathrm{d}t\int_0^{\infty}\exp(-2kh)\,j_n(\Delta)\,j_{n'}(\Delta)\,\mathrm{d}k, \tag{24}$$

$$H(n,m,n',m') = \int_{-\pi}^{\pi} T_n^m(t)\,T_{n'}^{m'}(t)\,\mathrm{d}t\int_L \frac{\exp(-2kh)}{k-\nu}\,j_n(\Delta)\,j_{n'}(\Delta)\,\mathrm{d}k. \tag{25}$$

The essential task in solving (23) is the evaluation of these two integrals. Since

$$j_n(\Delta) = \left(\frac{\pi}{2\Delta}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(\Delta), \tag{26}$$

where $J_{n+\frac{1}{2}}$ is the Bessel function of the first kind, and (Watson 1944)

$$J_{n+\frac{1}{2}}(\Delta)\,J_{n'+\frac{1}{2}}(\Delta) = \sum_{s=0}^{\infty}\frac{(-1)^s(\frac{1}{2}\Delta)^{n+n'+1+2s}\,\Gamma(n+n'+2s+2)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(n+s+\frac{3}{2})\,\Gamma(n'+s+\frac{3}{2})}, \tag{27}$$

where Γ is the gamma function, we obtain

$$\begin{aligned}
I'(n, n', t) &= \int_0^\infty \exp(-2kh) j_n(\Delta) j_{n'}(\Delta)\,\mathrm{d}k \\
&= \int_0^\infty \exp(-2kh) \frac{\pi}{2kc\cos t} \sum_{s=0}^{\infty} \frac{(-1)^s(\frac{1}{2}ck\cos t)^{n+n'+2s+1}\,\Gamma(n+n'+2s+2)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(n+s+\frac{3}{2})\,\Gamma(n'+s+\frac{3}{2})}\,\mathrm{d}k \\
&= \tfrac{1}{4}\pi \sum_{s=0}^{\infty} \frac{(-1)^s(\frac{1}{2}c\cos t)^{n+n'+2s}\,\Gamma(n+n'+2s+2)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(n+s+\frac{3}{2})\,\Gamma(n'+s+\frac{3}{2})} \int_0^\infty \exp(-2kh)\, k^{n+n'+2s}\,\mathrm{d}k \\
&= \tfrac{1}{4}\pi \sum_{s=0}^{\infty} \frac{(-1)^s(\frac{1}{2}c\cos t)^{n+n'+2s}\,\Gamma(n+n'+2s+2)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(n+s+\frac{3}{2})\,\Gamma(n'+s+\frac{3}{2})} \frac{\Gamma(n+n'+2s+1)}{(2h)^{n+n'+2s+1}} \\
&= \tfrac{1}{4}\pi(\tfrac{1}{2}c\cos t)^{n+n'} \frac{1}{(2h)^{n+n'+1}} \sum_{s=0}^{\infty} (-1)^s(\tfrac{1}{2}c\cos t)^{2s} \frac{1}{(2h)^{2s}} \\
&\qquad \times \frac{\Gamma(n+n'+2s+2)\,\Gamma(n+n'+2s+1)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(n+s+\frac{3}{2})\,\Gamma(n'+s+\frac{3}{2})}. \qquad (28)
\end{aligned}$$

Using Abramowitz & Stegun (1965)

$$\Gamma(n+s+\tfrac{3}{2}) = \frac{\Gamma(2n+2s+2)\,(2\pi)^{\frac{1}{2}}}{2^{2n+2s+2-\frac{1}{2}}\,\Gamma(n+s+1)}, \tag{29}$$

equation (28) becomes

$$\begin{aligned}
I'(n, n', t) &= \frac{1}{2h}\left(\frac{c\cos t}{h}\right)^{n+n'} \sum_{s=0}^{\infty} (-1)^s \left(\frac{c\cos t}{h}\right)^{2s} \\
&\qquad \times \frac{\Gamma(n+n'+2s+2)\,\Gamma(n+n'+2s+1)\,\Gamma(n+s+1)\,\Gamma(n'+s+1)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(2n+2s+2)\,\Gamma(2n'+2s+2)}. \qquad (30)
\end{aligned}$$

By substitution of $I'(n, n', t)$ into (24), $I(n, m, n', m')$ can be obtained in the form

$$I(n, m, n', m') = 4\int_0^{\frac{1}{2}\pi} T_n^m(t)\, T_{n'}^{m'}(t)\, I'(n, n', t)\,\mathrm{d}t, \tag{31a}$$

for $n+m+n'+m'$ even; and

$$I(n, m, n', m') = 0, \tag{31b}$$

for $n+m+n'+m'$ odd.

The series in (30) converges absolutely and can be accurately computed numerically when $c\cos t < h$. If $c\cos t \geqslant h$, the direct integration in (28) may be more efficient.

To calculate the integration over k in (25), we write

$$H(n, m, n', m') = \int_{-\pi}^{\pi} T_n^m(t)\, T_{n'}^{m'}(t)\, H'(n, n', t)\,\mathrm{d}t, \tag{32}$$

where

$$H'(n, n', t) = \int_L \frac{\exp(-2kh)}{k-\nu} j_n(\Delta) j_{n'}(\Delta)\,\mathrm{d}k. \tag{33}$$

The definition of L is given as before.

Substituting (26) and (27) into (33), and also using (29), we obtain

$$H'(n,n',t) = \tfrac{1}{4}\pi \sum_{s=0}^{\infty} \frac{(-1)^s(\tfrac{1}{2}c\cos t)^{n+n'+2s}\,\Gamma(n+n'+2s+2)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(n+s+\tfrac{3}{2})\,\Gamma(n'+s+\tfrac{3}{2})} H'^{2s+n+n'}(\nu)$$

$$= (2c\cos t)^{n+n'} \sum_{s=0}^{\infty} (-1)^s (2c\cos t)^{2s}$$

$$\times \frac{\Gamma(n+n'+2s+2)\,\Gamma(n+s+1)\,\Gamma(n'+s+1)}{s!\,\Gamma(n+n'+s+2)\,\Gamma(2n+2s+2)\,\Gamma(2n'+2s+2)} H'^{2s+n+n'}(\nu), \quad (34)$$

where

$$H'^{s+n+n'}(\nu) = \int_L \frac{k^{s+n+n'}}{k-\nu} \exp(-2kh)\,\mathrm{d}k$$

$$= \int_L \frac{k^{s+n+n'} - \nu k^{s+n+n'-1}}{k-\nu} \exp(-2kh)\,\mathrm{d}k + \nu H'^{s+n+n'-1}(\nu)$$

$$= \int_0^{\infty} k^{s+n+n'-1} \exp(-2kh)\,\mathrm{d}k + \nu H'^{s+n+n'-1}(\nu)$$

$$= \frac{\Gamma(s+n+n')}{(2h)^{s+n+n'}} + \nu H'^{s+n+n'-1}(\nu), \quad (35a)$$

$$H'^{0}(\nu) = \exp(-2\nu h)\,[-E_i(2\nu h) - \mathrm{i}\pi], \quad (35b)$$

and E_i is the exponential integral. As in the case of (30), (34) does not offer a practical computational method for large c and ν; but it is particularly efficient when $c\cos t < h$ and $2c\nu < 1$.

After $H'(n, n', t)$ has been found, $H(n, m, n', m')$ in (32) can be obtained in a similar manner to (31):

$$H(n,m,n',m') = 4\int_0^{\frac{1}{2}\pi} T_n^m(t)\,T_{n'}^{m'}(t)\,H'(n,n',t)\,\mathrm{d}t, \quad (36a)$$

for $n+m+n'+m'$ even;

$$H(n,m,n',m') = 0, \quad (36b)$$

for $n+m+n'+m'$ odd.

The formulations above provide a basis for obtaining the analytic solution for the scattering potential of a submerged spheroid in head or following seas. The remaining problem concerns the existence and uniqueness of a finite solution of the infinite sets of equations (23). A sufficient condition is that the sum of the moduli of the coefficients is finite (e.g. Hulme 1982). However, it is not unreasonable to assume that the solution exists and is unique, as stable and converged results for the wave resistance on the spheroid have been obtained by Farell (1973) using this method. Thus we may be able to find the solution directly without proof of its existence and uniqueness first.

The solution of equation (23) also requires calculation of the spherical Bessel function and the Legendre function. In the general case, this can be achieved by means of a numerical method based on recurrence relations (Abramowitz & Stegun 1965); but this procedure cannot always guarantee adequate accuracy for any given value of the argument. Round-off error at each step can accumulate a significant error in the summed result. Fortunately, we have found that the solution of equation (23) converges very rapidly as n increases, especially at low frequency. Thus we use explicit expressions for the spherical Bessel functions and Legendre functions up to $n = 5$; this number of terms has been found to be sufficient when $\nu a \leqslant 1$ to guarantee the accuracy of the numerical results given below.

3. The exciting force

In the linearized potential theory, the hydrodynamic pressure can be written as

$$p = -\mathrm{i}\omega\rho(\phi_\mathrm{i}+\phi_\mathrm{d}). \tag{37}$$

The exciting force on the spheroid is given by the integration of the pressure over its surface, or

$$\boldsymbol{F} = -\mathrm{i}\omega\rho\iint_{S_0}(\phi_\mathrm{i}+\phi_\mathrm{d})\,\boldsymbol{n}\,\mathrm{d}S. \tag{38}$$

Here $\boldsymbol{n}$ is the normal out of the body surface having the components

$$\boldsymbol{n} = (n_x, n_y, n_z), \tag{39}$$

where

$$n_x = \frac{b\cos\theta}{(b^2\cos^2\theta+a^2\sin^2\theta)^{\frac{1}{2}}}, \tag{40a}$$

$$n_y = \frac{a\sin\theta\sin\phi}{(b^2\cos^2\theta+a^2\sin^2\theta)^{\frac{1}{2}}}, \tag{40b}$$

$$n_z = \frac{a\sin\theta\cos\phi}{(b^2\cos^2\theta+a^2\sin^2\theta)^{\frac{1}{2}}}. \tag{40c}$$

Substituting (19), (21) and (40) into (38), we obtain the x-component of the exciting force

$$\begin{aligned}
F_x &= -\mathrm{i}\omega\rho\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi(\phi_\mathrm{i}+\phi_\mathrm{d})\,b^2\sin\theta\cos\theta \\
&= -\mathrm{i}\omega\pi\int_0^\pi d\theta\,2\pi\Bigg[\frac{\mathrm{i}g}{\omega}\exp(-\nu h)\sum_{n=0}^{\infty}D_{n,0}(\nu,\beta)\,P_n(\cosh\eta_0)\,P_n(\cos\theta) \\
&\quad -c\sum_{n=0}^{\infty}A_n^0\,Q_n(\cosh\eta_0)\,P_n(\cos\theta) \\
&\quad +\frac{c^2}{4\pi}\sum_{n=0}^{\infty}P_n(\cosh\eta_0)\,P_n(\cos\theta)\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n'+1}\,\mathrm{i}^{n'+m'}\,I(n,0,n',n') \\
&\quad +\frac{c^2\nu}{2\pi}\sum_{n=0}^{\infty}P_n(\cosh\eta_0)\,P_n(\cos\theta)\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n'+1}\,\mathrm{i}^{n'+m'} \\
&\quad \times H(n,0,n',m')\Bigg]\tfrac{1}{2}b^2\sin 2\theta \\
&= -\mathrm{i}\omega\rho\tfrac{4}{3}\pi b^2\Bigg\{\frac{\mathrm{i}g}{\omega}\exp(-\nu h)\,D_{1,0}(\nu,\beta)\,P_1(\cosh\eta_0)-cA_1^0\,Q_1(\cosh\eta_0) \\
&\quad +\frac{c^2}{4\pi}P_1(\cosh\eta_0)\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}A_{n'}^{m'}(-1)^{n'+1}\,\mathrm{i}^{n'+m'}\,[I(1,0,n',m')+2\nu H(1,0,n',m')]\Bigg\}.
\end{aligned} \tag{41}$$

Substituting (23) when $n = 1$, $m = 0$ into (41), we obtain an extremely simple result:

$$F_x = -\mathrm{i}\omega\rho\tfrac{4}{3}\pi b^2 \left\{ -cA_1^0 Q_1(\cosh\eta_0) + cA_1^0 \frac{\mathrm{d}}{\mathrm{d}\eta}[Q_1(\cosh\eta_0)] \frac{P_1(\cosh\eta_0)}{\mathrm{d}/\mathrm{d}\eta\,[P_1(\cosh\eta_0)]} \right\}$$

$$= \tfrac{4}{3}\pi\rho(a^2-b^2)^{\frac{3}{2}}\,\omega \mathrm{i} A_1^0. \tag{42a}$$

Since $\sin\phi$ is orthogonal to all terms $\cos m\phi$ ($m = 0, 1, \ldots$), we immediately have

$$F_y = 0, \tag{42b}$$

due to the symmetry of the problem. And, similar to (42a), we obtain

$$F_z = \tfrac{4}{3}\pi\rho(a^2-b^2)^{\frac{3}{2}}\,\omega \mathrm{i} A_1^1. \tag{42c}$$

4. The special case of a sphere

When $a = b = r_0$, the spheroid becomes a sphere. The case of a submerged sphere has been analysed by Wang (1986), using the method of multipole expansions. By adapting our source-distribution technique to this particular case, one can provide a comparison with his results. In the spheroidal coordinates defined in (1), the case of a sphere arises when $\eta_0 \to \infty$ and $c \to 0$ but

$$c\cosh\eta_0 = r_0. \tag{43}$$

In view of this limiting process, the problem needs a modification to the general procedure given above.

We first define

$$a_n^m = \frac{c(-1)^m Q_n^m(\cosh\eta_0)}{r_0(n+m)!} A_n^m. \tag{44}$$

Then by writing the associated Legendre functions involving η in terms of a large-argument asymptotic form, and using the small-argument form of the Bessel function in (26), an expression for ϕ_{d} may be obtained for the sphere which is analogous to (19) for the spheroid. This involves the new set of constants a_n^m, which are the solutions of an infinite set of linear algebraic equations obtained by satisfying the body-surface boundary condition. This set of equations is analogous to (23). Finally the exciting forces are obtained in a very similar form to (42), with a_1^m in place of A_1^m and r_0 in place of $(a^2-b^2)^{\frac{1}{2}}$.

5. Numerical results and discussion

The infinite sets of (23) and their counterparts for the sphere have been solved by truncating the series at a finite number $n = N$. Tables 1 and 2 show the convergence of the surge and heave forces with increase of N, for a submerged sphere at depth $h = 1.5a$. These have been expressed in the non-dimensional form $f = F/\tfrac{4}{3}\pi ab^2\rho g\nu\exp(-\nu h)$ (with $a = b$ for the sphere). The tables lead to the conclusion that over this range of non-dimensional frequency $N = 5$ gives accuracy to four significant digits (results for $N = 6$ being found identical to the fourth decimal point). The comparison with the results obtained by Wang (1986) shows that the difference at most happens at the fourth decimal point (with the exception of $\mathrm{Im}(f_x)$ when $\nu a \geqslant 0.6$).

At higher frequency than $\nu a = 1$ more terms of the series are required, but it is found that $N = 10$ is more than enough to obtain the converged results of table 3.

νa	$-\mathrm{Re}(f_x)$				$-\mathrm{Im}(f_x)$			
	$N=3$	$N=4$	$N=5$	Wang	$N=3$	$N=4$	$N=5$	Wang
0.1	0.0019	0.0019	0.0019	0.0019	1.5419	1.5419	1.5419	1.5422
0.2	0.0114	0.0114	0.0114	0.0114	1.5587	1.5587	1.5587	1.5591
0.3	0.0297	0.0297	0.0297	0.0298	1.5734	1.5734	1.5734	1.5739
0.4	0.0546	0.0546	0.0546	0.0546	1.5819	1.5820	1.5820	1.5827
0.5	0.0825	0.0825	0.0825	0.0826	1.5825	1.5827	1.5827	1.5835
0.6	0.1101	0.1102	0.1102	0.1103	1.5753	1.5755	1.5755	1.5766
0.7	0.1350	0.1352	0.1352	0.1354	1.5614	1.5618	1.5618	1.5630
0.8	0.1558	0.1561	1.1562	0.1564	1.5427	1.5433	1.5433	1.5446
0.9	0.1719	0.1724	0.1725	0.1728	1.5208	1.5217	1.5218	1.5231
1.0	0.1832	0.1842	0.1843	0.1846	1.4975	1.4986	1.4987	1.5001

TABLE 1. Convergence of the x-component of exciting force on a sphere ($h = 1.5a$)

νa	$\mathrm{Re}(f_z)$				$-\mathrm{Im}(f_z)$			
	$N=3$	$N=4$	$N=5$	Wang	$N=3$	$N=4$	$n=5$	Wang
0.1	1.5857	1.5858	1.5858	1.5860	0.0039	0.0039	0.0039	0.0039
0.2	1.6020	1.6203	1.6203	1.6205	0.0245	0.0246	0.0246	0.0247
0.3	1.6491	1.6492	1.6492	1.6496	0.0649	0.0649	0.0649	0.0652
0.4	1.6623	1.6625	0.6625	0.6628	0.1193	0.1193	0.1193	0.1197
0.5	1.6554	1.6557	1.6557	1.6558	0.1779	0.1780	0.1780	0.1786
0.6	1.6299	1.6304	1.6304	1.6304	0.2317	0.2319	0.2319	0.2325
0.7	1.5912	1.5919	1.5919	1.5918	0.2745	0.2749	0.2750	0.2755
0.8	1.5456	1.5464	1.5465	1.5462	0.3044	0.3051	0.3051	0.3055
0.9	1.4979	1.4990	1.4991	1.4988	0.3218	0.3230	0.3231	0.3234
1.0	1.4518	1.4530	1.4531	1.4530	0.3290	0.3307	0.3309	0.3310

TABLE 2. Convergence of the z-component of exciting force on a sphere ($h = 1.5a$)

νa	$-\mathrm{Re}(f_x)$		$-\mathrm{Im}(f_x)$		$\mathrm{Re}(f_z)$		$-\mathrm{Im}(f_z)$	
	Present	Wang	Present	Wang	Present	Wang	Present	Wang
2.0	0.1614	0.1614	1.3130	1.3124	1.1903	1.1903	0.2182	0.2182
3.0	0.0902	0.0899	1.2239	1.2193	1.1004	1.1004	0.1083	0.1084
4.0	0.0433	0.0430	1.1585	1.1509	1.0311	1.0313	0.0485	0.0485
5.0	0.0185	0.0184	1.0765	1.0664	0.9358	1.0643	0.0195	0.0196

TABLE 3. The exciting forces on a sphere at high frequencies ($h = 1.5a$)

The difference between the present results and Wang's is believed to be associated with the method of constructing the infinite set of linear equations based on the body-surface condition. We have used the orthogonality of $P_1^m(x)$ and $P_n^m(x)$ over the range $[-1, 1]$ (Abramowitz & Stegun 1965 equation (8.14.11)), while Wang imposed the body-surface condition at 34 discretized points having values of θ in $P_n^m(\cos\theta)$ evenly distributed between 0 and π.

Tables 4 to 6 give the results of the non-dimensional exciting force on a spheroid

νa	$-\mathrm{Re}\,(f_x)$			$-\mathrm{Im}\,(f_x)$		
	$N=3$	$N=4$	$N=5$	$N=3$	$N=4$	$N=5$
0.1	0.0000	0.0000	0.0000	1.0542	1.0542	1.0542
0.2	0.0002	0.0002	0.0002	1.0536	1.0536	1.0536
0.3	0.0007	0.0007	0.0007	1.0514	1.0514	1.0514
0.4	0.0017	0.0017	0.0017	1.0475	1.0475	1.0475
0.5	0.0031	0.0031	0.0031	1.0420	1.0420	1.0420
0.6	0.0051	0.0051	0.0051	1.0347	1.0348	1.0348
0.7	0.0077	0.0077	0.0077	1.0257	1.0258	1.0258
0.8	0.0110	0.0110	0.0110	1.0149	1.0151	1.0151
0.9	0.0151	0.0151	0.0151	1.0020	1.0022	1.0022
1.0	0.0198	0.0198	0.0198	0.9873	0.9875	0.9876

TABLE 4. Convergence of the x-component of exciting force on a spheroid ($h = 2b$, $a = 6b$)

νa	$\mathrm{Re}\,(f_z)$			$-\mathrm{Im}\,(f_z)$		
	$N=3$	$N=4$	$N=5$	$N=3$	$N=4$	$N=5$
0.1	2.0255	2.0256	2.0256	0.0002	0.0002	0.0002
0.2	2.0344	2.0344	2.0344	0.0017	0.0017	0.0017
0.3	2.0415	2.0416	2.0416	0.0054	0.0054	0.0054
0.4	2.0465	2.0466	2.0466	0.0121	0.0121	0.0121
0.5	2.0486	2.0487	2.0487	0.0222	0.0222	0.0222
0.6	2.0469	2.0470	2.0471	0.0361	0.0361	0.0361
0.7	2.0406	2.0408	2.0408	0.0538	0.0538	0.0538
0.8	2.0297	2.0300	2.0301	0.0751	0.0752	0.0752
0.9	2.0111	2.0115	2.0116	0.0995	0.0995	0.0995
1.0	1.9865	1.9870	1.9871	0.1264	0.1265	0.1265

TABLE 5. Convergence of the z-component of exciting force on a spheroid ($h = 2b$, $a = 6b$)

νa	$-\mathrm{Re}\,(f_x)$	$-\mathrm{Im}\,(f_x)$	$\mathrm{Re}\,(f_z)$	$-\mathrm{Im}\,(f_z)$
1.5	0.0528	0.8832	1.7580	0.2690
2.0	0.0903	0.7267	1.3913	0.3553
2.5	0.1109	0.5334	0.9834	0.3579
3.0	0.1045	0.3367	0.6034	0.3043
3.5	0.0793	0.1662	0.2851	0.2247
4.0	0.0486	0.0356	0.0453	0.1397
4.5	0.0214	−0.0519	−0.1101	0.0670
5.0	0.0010	−0.0995	−0.1904	0.0139

TABLE 6. The exciting forces on a spheroid at high frequencies ($h = 2b$, $a = 6b$)

of aspect ratio $a = 6b$, submerged at depth $h = 2b$. The results again show that $N = 5$ is enough when $\nu a \leqslant 1.0$. When $\nu a > 1.0$, we need more terms but $N = 8$ can give the converged results in table 6. We have also investigated several other cases, and the results are given in figure 1 to figure 4. The general feature of these figures is that the non-dimensional force is reduced by an increase of either the submergence or the length of the major axis for a given length of the minor axis.

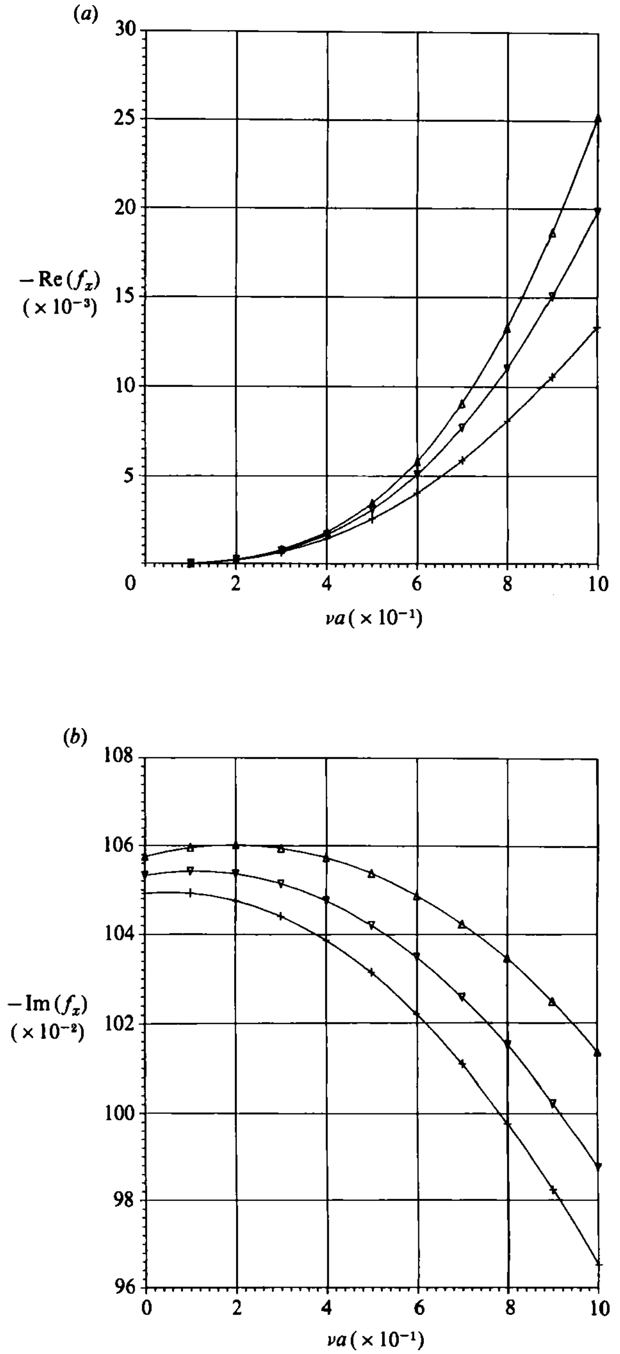
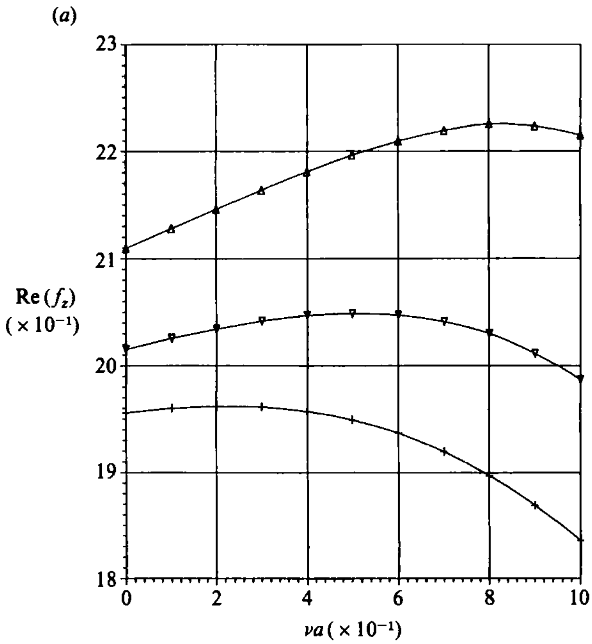


FIGURE 1. Non-dimensional surge force on a spheroid ($a = 6b$) at different submergences ($\triangle$, $h = 1.5b$; ∇, $h = 2.0b$; $+$, $h = 3.0b$). (a) Real part; (b) Imaginary part.



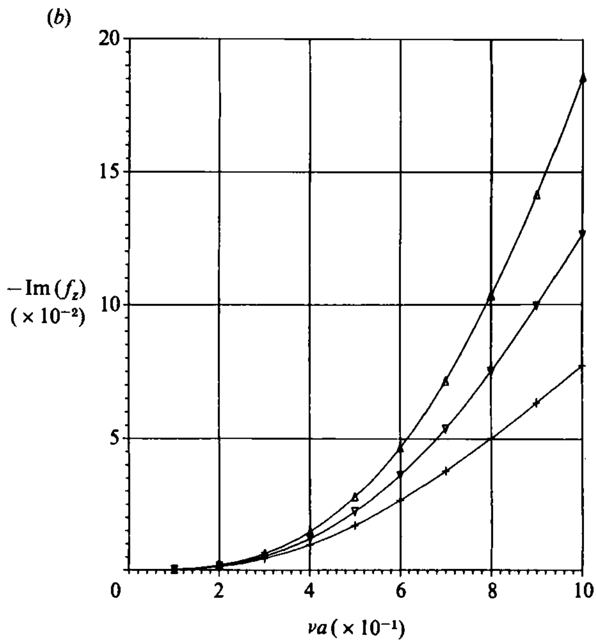


FIGURE 2. Non-dimensional heave force on a spheroid ($a = 6b$) at different submergences (△, $h = 1.5b$; ▽, $h = 2.0b$; +, $h = 3.0b$). (*a*) Real part; (*b*) Imaginary part.

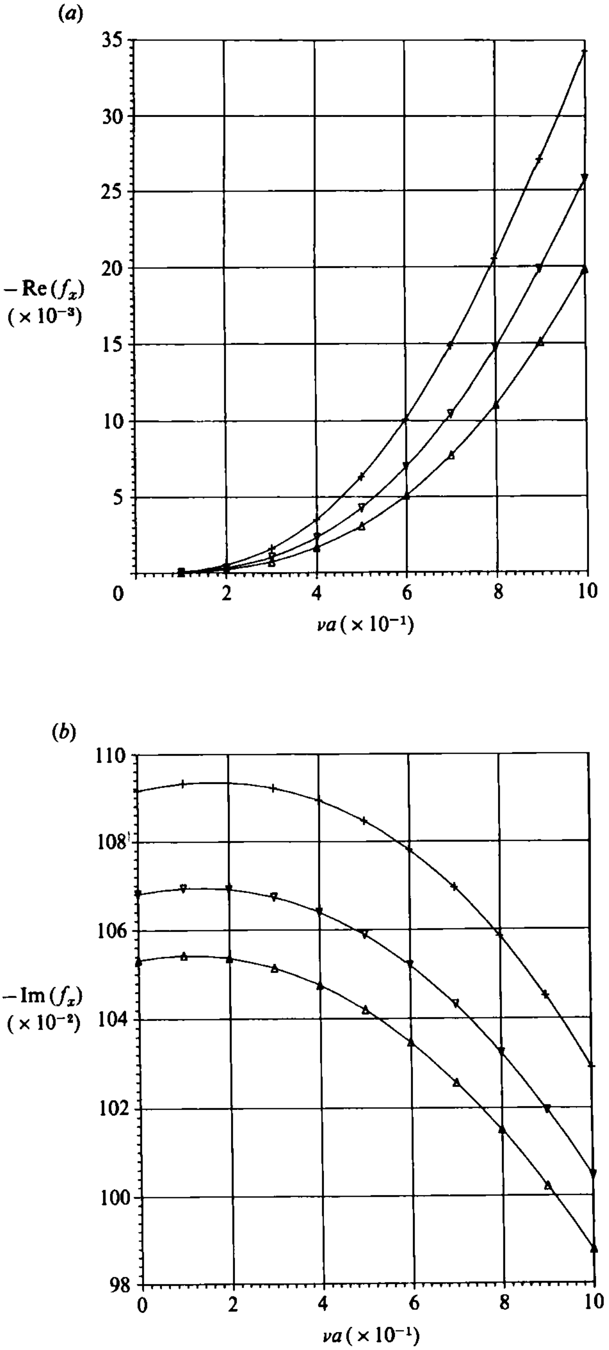


FIGURE 3. Non-dimensional surge force on spheroids of different aspect ratios ($h = 2b$; $\triangle$, $a = 6b$; ∇, $a = 5b$; $+$, $a = 4b$). (*a*) Real part; (*b*) Imaginary part.

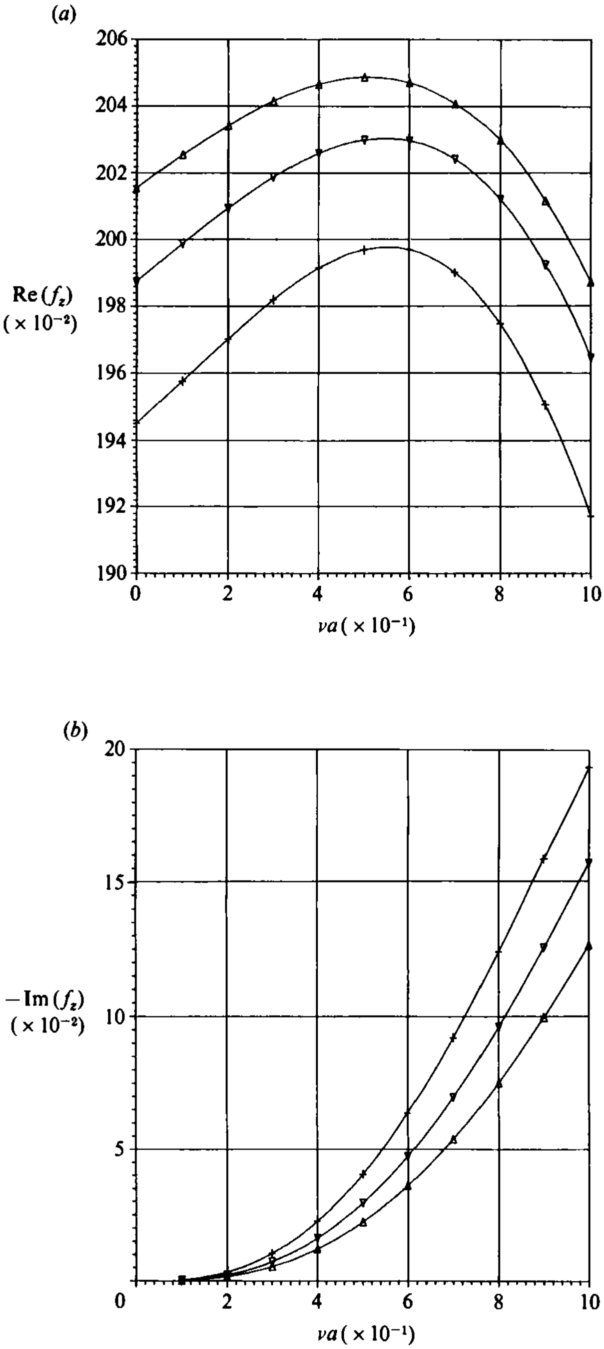


FIGURE 4. Non-dimensional heave force on spheroids of different aspect ratios ($h = 2b$; $\triangle$, $a = 6b$; ∇, $a = 5b$; $+$, $a = 4b$). (*a*) Real part; (*b*) Imaginary part.

6. Concluding remarks

This work is motivated by the need for an analytic solution to the problem of a body advancing in waves. The method which is commonly used at the present time is the simplified two-dimensional strip theory, which requires the body to be slender. Application of strip theory is also limited by the assumption regarding the relative magnitude of forward speed and encounter frequency. Recently there have been some attempts to obtain results using three-dimensional numerical methods, either by means of singularity distributions (Chang 1977; Inglis & Price 1981; Kobayashi 1981; Guevel & Bougis 1982), or by combining localized finite elements with a boundary integral equation (Wu 1986). Since it has been observed that stable and converged results are not easy to obtain and the computations are rather expensive, extensive numerical investigation has not so far been achieved. It is therefore necessary to provide some analytical solutions as a basis for checking the numerical results. It may be seen that the present approach can be readily extended to the problem of a submerged spheroid at forward speed in waves, by substituting the corresponding Green function into (7) (Wu 1986).

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